

State the formal definition of "horizontal asymptote".

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f HAS A HORIZONTAL ASYMPTOTE $y = b$ IF ②

$$\lim_{x \rightarrow \infty} f(x) = b \text{ OR } \lim_{x \rightarrow -\infty} f(x) = b$$

③

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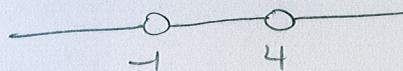
Let $f(x) = \frac{1-x^2}{x^2 - 3x - 4}$.

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- [a] Find all intervals on which f is continuous.

$1-x^2, x^2-3x-4$ ARE CONT ON \mathbb{R}

$x^2-3x-4 \neq 0$ IF $(x-4)(x+1) \neq 0$ $\Leftrightarrow x \neq 4, -1$ (3)



$$(-\infty, -1), (-1, 4), (4, \infty)$$

(3) (3) (3)

- [b] Find the limit of f at each discontinuity.

Each limit should be a number, ∞ or $-\infty$. Write DNE only if the other possibilities do not apply.

$$\begin{aligned} & \lim_{x \rightarrow -1} \frac{1-x^2}{x^2-3x-4} \\ &= \lim_{x \rightarrow -1} \frac{(1-x)(1+x)}{(x+1)(x-4)} \\ &= \boxed{\lim_{x \rightarrow -1} \frac{1-x}{x-4}} \quad (3) \\ &= \frac{1-(-1)}{-1-4} = \boxed{-\frac{2}{5}} \quad (2) \\ & \lim_{x \rightarrow 4^-} \frac{1-x}{x-4} = \infty \quad \frac{-3}{0^-} \quad (4) \\ & \lim_{x \rightarrow 4^+} \frac{1-x}{x-4} = -\infty \quad \frac{-3}{0^+} \quad (4) \\ & \text{so } \lim_{x \rightarrow 4} \frac{1-x^2}{x^2-3x-4} \text{ DNE} \quad (2) \end{aligned}$$

- [c] State the type of each discontinuity in [b].

(3) $x = -1$ REMOVABLE DISCONTINUITY

(3) $x = 4$ INFINITE DISCONTINUITY

- [d] Find the equations of all horizontal asymptotes of f .

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{1-x^2}{x^2-3x-4} \cdot \boxed{\frac{\frac{1}{x^2}}{\frac{1}{x^2}}} \quad (2) \\ &= \boxed{\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}-1}{1-\frac{3}{x}-\frac{4}{x^2}}} \quad (3) \\ &= \frac{0-1}{1-0-0} = \boxed{-1} \quad (2) \\ & \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}-1}{1-\frac{3}{x}-\frac{4}{x^2}} \\ &= \frac{0-1}{1-0-0} = \boxed{-1} \quad (2) \end{aligned}$$

HORIZONTAL ASYMPTOTE

(3) $y = -1$

If $f(x) = \sqrt{5-x^2}$, find $f'(x)$.

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$$\boxed{\lim_{h \rightarrow 0} \frac{\sqrt{5-(x+h)^2} - \sqrt{5-x^2}}{h}} \quad \textcircled{4}$$
$$\cdot \frac{\sqrt{5-(x+h)^2} + \sqrt{5-x^2}}{\sqrt{5-(x+h)^2} + \sqrt{5-x^2}}$$

$$= \lim_{h \rightarrow 0} \frac{5-(x^2+2xh+h^2)-(5-x^2)}{h(\sqrt{5-(x+h)^2} + \sqrt{5-x^2})}$$

$$= \boxed{\lim_{h \rightarrow 0} \frac{-2xh-h^2}{h(\sqrt{5-(x+h)^2} + \sqrt{5-x^2})}} \quad \textcircled{6}$$

$$= \boxed{\lim_{h \rightarrow 0} \frac{-2x-h}{\sqrt{5-(x+h)^2} + \sqrt{5-x^2}}} \quad \textcircled{4}$$

$$\textcircled{4} \boxed{\frac{-2x}{2\sqrt{5-x^2}}} = \boxed{\frac{-x}{\sqrt{5-x^2}}} \quad \textcircled{2}$$

State the Intermediate Value Theorem.

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IF f is continuous on (a, b) , 2⁺

AND d is between $f(a)$ and $f(b)$, 2⁺

THEN $d = f(c)$ for some $c \in (a, b)$, 5

Find the equation of the tangent line to the graph of $f(x) = 4x^2 - x^3$ at the point where $x = 2$.

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$$\lim_{b \rightarrow 2} \frac{f(b) - f(2)}{b - 2}$$

$$= \boxed{\lim_{b \rightarrow 2} \frac{4b^2 - b^3 - 8}{b - 2}} \text{ (6)}$$

$$\begin{array}{r} 2 \\[-4pt] | -1 \quad 4 \quad 0 \quad -8 \\[-4pt] \underline{-}2 \quad 4 \quad 8 \\[-4pt] -1 \quad 2 \quad 4 \quad | 0 \end{array}$$

$$= \boxed{\lim_{b \rightarrow 2} (-b^2 + 2b + 4)} \text{ (6)}$$

$$= -4 + 4 + 4 = \boxed{4} \text{ (4)}$$

$$\boxed{y - 8 = 4(x - 2)} \text{ (4)}$$

Prove that the equation $2x^2 = 1 + e^x$ has a solution in the interval $(-1, 1)$.

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$$2x^2 - e^x = 1$$

LET $f(x) = 2x^2 - e^x$

f is continuous on \mathbb{R} since $2x^2, e^x$ are both continuous on \mathbb{R} (1/2)
AND SO IS THEIR DIFFERENCE. (1/2)

$$f(-1) = 2 - e^{-1} = 2 - \frac{1}{e} \approx 2 - \frac{1}{2.7} > 1$$

$$f(1) = 2 - e \approx 2 - 2.7 < 0$$

$$f(1) < 1 < f(-1)$$

② EACH EXCEPT AS NOTED

BY IVT, $f(x) = 2x^2 - e^x = 1$ FOR SOME $x \in (-1, 1)$

i.e. $2x^2 = 1 + e^x$

Find a function f and a non-zero number a such that the derivative of f at a is given by

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$$\lim_{h \rightarrow 0} \frac{e^{1-h} - e}{h} = \lim_{h \rightarrow 0} \frac{e^{-(-1+h)} - e}{h} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Show that your answers are correct using the definition of the derivative at a point.

$$f(x) = e^{-x} \quad (4)$$

$$a = -1 \quad (4)$$

$$\left| \lim_{h \rightarrow 0} \frac{e^{-(-1+h)} - e^{-(-1)}}{h} \right| \quad (4)$$

$$= \left| \lim_{h \rightarrow 0} \frac{e^{1-h} - e}{h} \right| \quad (3)$$

The graph of f is shown to the right. Arrange the following from least (most negative) to greatest (most positive). **SCORE:** _____ / 15 PTS

$$f'(0)$$

$$f'(-1)$$

$$f'(-2)$$

$$f'(2)$$

$$f'(3)$$

$$\frac{f'(3)}{\text{LEAST}} < \frac{f'(0)}{\text{ }} < \frac{f'(-2)}{\text{ }} < \frac{f'(2)}{\text{ }} < \frac{f'(-1)}{\text{GREATEST}}$$

